The Definition Of Chaos

Chaos is a concept that permeates into our lives from our heartbeats to the fish population in the reflecting pond. To many this concept strikes fear in their hearts because this means everything in the universe is unordered. This a common misconception that chaos is not without order.

Section 1: The definition of Chaos

There are many different concepts of chaos so before devling into the topics it is necessary that chaos first be defined. “Chaos is the study of complex nonlinear dynamic systems.” Nonlinear dynamic systems are systems that contain a variable with an exponent other than one (Dicionary.com). An informal way of looking at nonlinearity is when “the act of playing the game has a way of changing the rules” (Gleick 24). “Chaos is the study of forever changing complex systems based on mathematical concepts of recursion” (Gleick 306). An “informal mathematical definition” of chaos is “effectively unpredictable long time behavior arising in a deterministic dynamical system because of sensitivity to initial conditions” (Donnelly 5).

“A dynamical system is chaotic if it: has a dense collection of points with periodic orbits, is sensitive to the initial condition of the system in which initially nearby points can evolve quickly into very different states, and topologically transitive” (Weisstein Chaos). “ A function f is topologically transitive if, given any two intervals U and V, there is some positive integer k such that where denotes the iterate of” (Weisstein topologically transitive). Topogically transitive basically means points that are located in close proximity to each other will at some point in time get flung out to "big" sets, not necessarily sticking together in one concentrated cluster. There are multiple pathways of disorder yet only a few occur in nature. “The study of the movement of a system from order to chaos, is in a sense, the study of how a simple and limited motion breaks down so that nature begins to explore all the implications of the much larger phase space at it’s disposal” (Briggs 33). Examples of chaotic systems are population growth, epidemics to arrhythmic heart palpitations, and drippy facets. The concept of chaos and sensitivity to initial condition was understood centuries ago as shown by the following poem:

For want of a nail, the shoe was lost;

For want of a shoe, the horse was lost;

For want of a horse, the rider was lost;

For want of a rider, a message was lost;

For want of a message the battle was lost;

For want of a battle, the kingdom was lost! (Gleick 23).

Section 2: The Discovery of Chaos

Henri Poincare was the first person to conceptualize chaos. He wondered about the stability of the solar system. Isaac Newton had derived the shape of orbits mathematically using calculus. He assumed that the gravitational force is the inverse square of distance. Newton's inverse square law was well understood and accurately measured in Poincares' time. There was a small difficulty with the equations themselves, when changing from two to three bodies in Newton's equations caused the equations to become unsolvable. To solve the equations, one would have to make a series of approximations. These approximations work now, but in 10,000,000 years from now their accuracy may be tainted by the approximations (Briggs 28). " Until Poincare, chaos had been assumed to be entropic, inevitable and steady deterioration of a system or society" (Briggs 28). The deterioration was thought to be caused by external contingencies and functions of an outside system. "Now it appeared that a system sealed in a box and left untouched for billions of years could at any moment develop its own instabilities and chaos" (Briggs 28).

Section 3: Lorenz's Discovery

Edward Lorenz found that systems were sensitive to initial conditions. Lorenz was a meteorologist that was studying the weather. One day he wanted to restart a simulation, therefore entered into the computer numbers from an earlier printout of data. He then let the computer run the data again. When he came back, the outcome of the data was completely different the first set of data. The computer printed the results. It printed only three decimal places instead of six in order to save paper, which caused the varying outcomes. The tiny error introduced by dropping the digits after the thousandths place over time, grew to be an error as large as the range of possible solutions to the system (Gleick 14-18). "This implied that if the real atmosphere behaved in this method then we simply couldn't make forecasts two months ahead. The small errors in observation would amplify until they became large"(Elert Strange attractors). Lorenz's work remained unstudied for a long period of time, because Lorenz published his work in a small meteorological magazine. This sensitive dependence on initial conditions became known as the Butterfly Effect (Gleick 14-18). A paper was titled "Can the flap of a butterfly's wings in Brazil stir up a tornado in Texas?" The real question was can very small influences lead in due time to very big changes?" It's apparently because of the title of this paper that it's become known as the butterfly effect (Elert Strange Attractors). An essential characteristic of a chaotic system is sensitive dependence on initial conditions.

Sensitive dependence on initial conditions is illustrated in the Lorenz' Equations (Elert Strange Attractors). Nonlinearity and iteration are what make a system sensitive dependent to initial conditions.

Iteration is sometimes called the baker transformation because it is like a baker kneading dough. In both the iterative function and dough an action is performed upon the object to change it but using the old information. Think of when a bread is kneaded. First the dough is stretched and then folded back on to itself. That is the same as iteration an operation such as multiplication is used to get the next value which is then used to find the next value. If one watches one molecule in the bread that is being kneaded one can not predict where the molecule next to it will be in relation to the molecule at the end. The same can be said for chaos (Briggs 71).

The system of equations Lorenz used can exhibit non-chaotic behaviors. Given

the right initial conditions, some orbits settle down to a fixed value. These solutions could not possibly represent any real weather system, so there in turn, these solutions are less stressed. The most interesting solutions to the Lorenz system are the ergodic ones. Ergodic solutions are those solutions that visit almost every point in some region or those that will eventually approach an attractor. Real weather changes never settle down and also never repeat. There are days that are similar but never identical. The Lorenz model captures that concept. His model contains outcomes that are similar, yet they never repeat themselves (Elert Strange Attractors).

Weather has boundaries, there are limits to the kinds of weather that can be produced barring any changes to the earth, sun, and atmosphere. “The tropics can get hot, but they'll never get hot enough to melt lead. Storm winds may be stiff, but they'll never exceed the speed of sound” (Elert Strange Attractors). The Lorenz attractor is indeed bounded. The shape to which all orbits converge looks similar to the figurative butterfly Lorenz presented in Texas. While the attractor appears as two warped disks, their actual structure is far more complex, in that they are both ergodic and aperiodic, never repeating (Elert Strange Attractors).

Godel found an iterative paradox, the (w)hole in the center of our own logic, the potential chaos of the missing information, applies to many if not most of the things we think about. Godel proved that there will always be statements that are true that are unproveable. “He did this by proving a mathematical statement that said ‘This statement is unproveable’” (Briggs 76).

Section 4: Attractors

According to chaos theory there is an underlying order to systems. This underlying order can be shown in attractors. An attractor is “a state, into which a system settles over time” (Donnelly 27). Attractors have the important property of stability. In a real system motion tends to return to the attractor (Gleick 138). If system begins in close proximity to a state then that system will always be attracted to this state after enough iteration. In chaos, the attractors that are encountered are called strange attractors. “Strange attractor is an attraction set that has zero measure in the embedding phase space and has a fractal dimension. A state to which a system is attracted (under appropriate initial conditions) but to which it never settles down" (Weisstein Strange Attractor).

The strange attractor can be study through iterate systems. Iteration, feedback, is where the output of an equation is feed back into the system for the next output. Feedback can send a system to chaos or to stability. A microphone is an example of where feedback can send a system in to chaos. When standing too close to the speakers with a microphone a high pitch squeal is produced. That squeal is then fed through the microphone, through the speaker, back to the microphone and so on. This process amplifies the sound. A thermostat is an example where feedback sends a system to stability. When the thermostat reads a too low temperature, the furnace then produces heat, until; the air reaches a certain temperature. Then thermostat reads the appropriate temperature, the furnace shuts off. Stability is produced by the iteration of the furnace and the thermometer.

In feeding information back into mapping repeatedly one can see an example of this iteration.

An example of a mapping that illustrates attracting and repelling fixed points is draws all points asymptotically towards the origin while drives them away to infinity. The mapping will set every point on its own four-cycle around the origin. In higher dimensions attraction and repulsion are not limited to points. Attractors and repellers can form paths, surfaces, volumes, and their higher dimensional analogs. For example, the two-dimensional map attracts all points asymptotically to the x-axis. Likewise, a two-dimensional object can act as a repeller. Such is the case for the map (Elert Strange Attractors).

The Hénon attractor and the Ikeda attractor show fine structures that are infinite. All of the strange attractors have their origin in the study of real or idealized physical systems.

The study of perturbations in asteroid orbits gave rise to the Hénon's attractor and from a nonlinear optical system came Ikeda's attractor. "The systems studied by Hénon and Ikeda were two dimensional, but there is no reason why physical systems or the strange attractors that may arise from them should be limited to two dimensions." Here are some three dimensional strange attractors that arise from real or idealized physical systems (Elert Strange Attractors).

Chua (electronic circuit)

Duffing (nonlinear oscillator)

Lorenz (atmospheric convection)

Rössler (chemical kinetics)

One of the most famous strange attractor is the Lorenz attractor. The Lorenz attractor is a three dimensional object that bear a resemblance to a butterfly or a mask.

Named for Edward N. Lorenz ,who discovered it, the attractor came from a mathematical model of weather. A simple description of our atmosphere is “a rectangular slice of air heated from below and cooled from above by edges kept at constant temperatures. The bottom is heated by the earth and the top is cooled by the void of outer space “ (Elert Strange Attractors). As children learn in science classes warm air rises and cool air sinks. The transferring of heat from bottom to top causes a convection cell to be created. Three time-evolving variables can describe this model of the atmosphere:

"x" the convective flow

"y" the horizontal temperature distribution

"z" the vertical temperature distribution

with three parameters describing the character of the model itself

"s" [sigma] the ratio of viscosity to thermal conductivity

"r" [rho] the temperature difference between the top and bottom of the slice

"b" [beta] the width to height ratio of the slice

and three ordinary differential equations describing the appropriate laws of fluid dynamics

$dx/dt = s(y-x)$

$dy/dt = rx - y - xz$

$dz/dt = xy - bz$

Sensitive dependence is the first characteristic of chaos and fractals are the second. The Lorenz attractor is an example of a fractal. “A geometric figure of this sort with an infinite level of detail is called a fractal. Chaos always results in the formation of a

fractal, but not all fractals are associated with chaos.” One could claim that “chaos makes fractals” (Elert Strange Attractors).

Section 5: Bifurcation

“In a dynamical system, a bifurcation is a period doubling, quadrupling, etc., that accompanies the onset of chaos” (Weisstein Bifurcation) Bifurcation in a dynamic system is when a parameter is varied producing different solutions. This picture is a bifurcation when parameter r of a logistics map is varied.

The bifurcation diagram is a fractal. A fractal is something that is self-similar at different resolutions (Weinstein bifurcation).

“A more intuitive approach to orbits can be done through graphical representation using the following rules” (Elert Universality).

Draw a parabola of form and a line of form .

Draw both curves on the same axises. Pick a point on the x-axis. This point is our seed, starting value to plug in.

Draw a vertical straight line from the point until you intercept the parabola.

Draw a horizontal straight line from the intercept until you reach the diagonal line.

Repeat step 2 with this new point.

For the following diagrams zero will be the seed. This is because zero is “well-behaved”. The graph below “shows the simple fixed-point attractive behavior of the parameter value c = 1/4 for the seed value of 0.” The orbit moves towards ½ where an asymptote exisits (Elert Universality).

For the graph below the parameter c was set at -3/4. After more than 1000 iterations the orbit has not reached it’s final value because there is still a visible hole in the center (Elert Universality).

When c = -13/16 the orbit settles into a two-cycle, alternating between -3/4 and -1/4

(Elert Universality).

This diagram with c = -1.3 settles down to a four-cycle oscillation over the values -1.2996224637, 0.3890185483, -1.1486645691, and 0.0194302923 (Elert Universality).

The orbit produced by c=-1.4 has a period of 32 while our diagram below with c = -1.4015 has an orbit that never repeats. This shows that slight alterations in initial conditions can drastically change the outcomes of a system (Elert Universality).

At c = -1.8, the orbit covers every region of some subinterval of [-2, 2] (Elert bifurcation).

Section 6: Universality

Stated earlier in this paper, subregions within the bifurcation diagram look self-similar at different magnifications. “Such behavior is characteristic of geometric entities called fractals and is quite common in iterated mappings” (Elert Universality). In the period-doubling region, the distance between consecutive bifurcation points telescopes geometrically producing a ratio of the intervals approaches a constant value as the amount of bifurcations points approaches zero. This constant, Feigenbaum’s number, is found in many self-similar figures and has an approximate value of

4.66920160910299067185320382046620161725818557747576863274565134300413
43302113147371386897440239480138171659848551898151344086271420279325223
12442988890890859944935463236713411532481714219947455644365823793202009
5610583305754586176522220703854106467494942849814533917262005687556659
52339875603825637225 (Elert Universality).

Feigenbaum's constant and other characteristics of the bifurcation diagram show up in other figures. “In fact, iterating an infinite variety of iterative functions can generate amazingly similar bifurcation diagrams. Any function with a local maximum will produce a bifurcation diagram with period-doublings whose ratios approach the Feigenbaum number” (Elert Universality).

Feigenbaum’s number showed that there is universality in mathematics. This universality predicts the behavior of the logistic map and the quadratic map which are typical for many dynamical systems. Feigenbaum constant appeared in “an unruly mess of equations used to describe hydrodynamic flow”(Donnelly 34).

Feigenbaum's constants were originally derived from a mathematical model of animal populations.

In the segmented, fragmented world of modern science hydrodynamicists and population biologist rarely interact with one another. The realization that a set of five coupled differential equations describing turbulence could exhibit the same fundamental behavior as the one-dimensional map of the parabola on to itself was one of the key events in the history of mathematics (Elert Universality).

“Universality meant that different systems would behave identically” (Gleick 180). It also helped scientist realize that there is only a small amount of information that is important. Similar to an artist painting a tree, the artist looks at the whole tree. Then the artist paints enough detail so an observer can tell it is a tree but the artist does not paint every leaf. Similarly scientist can take a look at a system and describe the system with out describing every part of that system (Gleick 186).

Conclusion

The way nature has of coupling continuously changing things together creates systems that effectively resist change (Briggs 37). Chaos theory is the underlying order in the universe. “On a philosophical level, chaos theory may hold comfort for anyone who feels his or her place in the cosmos is inconsequential. Inconsequential sequences have a huge effect in a non-linear universe” (Briggs 75).